pulse shape. Near-optimal closed-loop mechanization also needs to be investigated.

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Absolute Stability of Symmetric Highly Maneuverable Missiles

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Introduction

THE design of cruciform missiles has been based on the use of three independent channels for roll pitch and yaw control. This basic design is based on the assumption that the interaction among these channels is low. This assumption is valid if the system works at low angles of attack. When higher angles of attack are required for increased maneuverability, the coupling terms become significantly dominant and the channel independence assumption is no longer valid. A different approach is then required for flight control system design.

This basic need was already identified by different researchers in this field. In particular, Ref. 1 is worth mentioning, where, using the model in the performance index concept, a coupled auto-pilot control was defined. The main limitation of this interesting work is to be found in the model used for analysis. As the authors themselves indicate, nonlinear effects were neglected and only the roll yaw dynamics were considered. Also, the pitch channel was assumed to be uncoupled and independently controlled. In Ref. 2, the roll nonlinear effects were explicity considered, but pitch yaw dynamics were completely neglected.

A full understanding of the open loop system behavior is a basic requirement in control design before an attempt at synthesizing a control policy can be made.

As is well known,[3,4] the combined pitch yaw roll dynamics of an aerodynamically controlled missile is highly complicated. The system model is necessarily of high order and strongly nonlinear.

Received Aug. 9, 1984; revision received Nov. 19, 1984.

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The purpose of the present work is to, first, define a system model, which is tractable in analytic terms, for a cruciform symmetric highly maneuverable missile. Next, based on this model, study the system behavior, and, in particular, find the conditions under which the open loop system is absolutely stable.

System Definition

In Fig. 1, a cruciform symmetric vehicle in atmospheric flight is depicted. The vehicle is aerodynamically controlled. In order to mathematically represent the angular motion of the vehicle at least eight state variables are required: p,q,r are the body rates in body coordinates; α,β, the angles of attack and sideslip, positive nose up and left, respectively, required to define the aerodynamic forces and moments; and ψ,θ,ϕ, the body angles with respect to inertial coordinates.

The differential equations relating these state variables are highly nonlinear. Realistic assumptions will be made in order to arrive at a system tractable in analytical terms. These assumptions are as follows:

1) Gravity effects are neglected. This assumption is valid for the highly maneuverable vehicles, here considered, capable of applying tens of g's in an arbitrary direction.

2) The moment of inertia matrix is of the form: $I_{ij}=0$ if $i\neq j$, $I_x \ll I_y = I_z$.

3) $\sin\alpha \sim \alpha$, $\cos\alpha \sim 1$; $\sin\beta \sim \beta$, $\cos\beta \sim 1$.

4) The aerodynamic pitch and yaw forces and moments in body axes are linear functions of α and β.

5) The control surface moments can be considered separately from body moments.

6) The control surfaces forces and the body aerodynamic side (out of maneuver plane) forces and moments can be neglected.[5]

With these assumptions the system equations for p,q,r,α,β, using the flight path axes[3,4] are

$$\dot{p} = L - L_p p + L_\delta \tag{1}$$

$$\dot{q} = -M_q q + M_\alpha \alpha + pr + M_\delta \tag{2}$$

$$\dot{r} = -N_r r - N_\beta \beta - qp + N_\delta \tag{3}$$

$$\dot{\alpha} = -Z_\alpha \alpha - \beta p - q \tag{4}$$

$$\dot{\beta} = -Y_\beta \beta + \alpha p + r \tag{5}$$

These five equations relating p,q,r,α,β, are independent of the three additional state variables ψ,θ,ϕ. Furthermore, the entire open loop system anagular dynamic behavior and stability is defined by Eqs. (1-5).

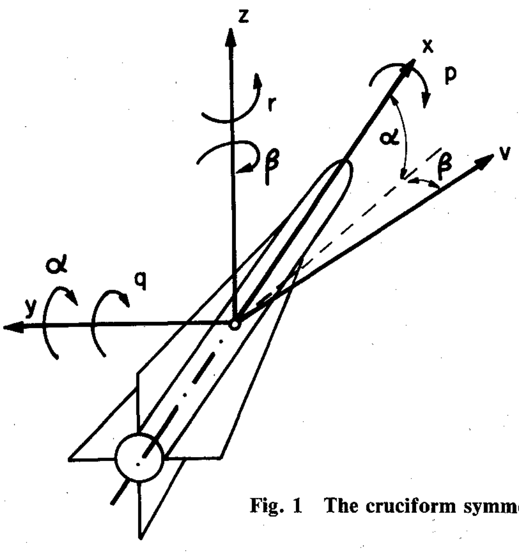

Fig. 1 The cruciform symmetric missile.

In Eqs. (1-5), L_δ, M_δ, N_δ are control-moments in roll pitch and yaw, and L_p, M_q, N_r, N_β, Z_α, Y_β are the system coefficients. Their values are defined by the aerodynamic coefficients in body axes, the flight conditions (height, velocity) and the body constants (mass, inertia). For constant flight conditions these system coefficients are very nearly constant. Furthermore, in comparison to the body dynamics, the flight conditions behave, in general, as slow functions fo time. In consequence, for the purposes of system stability analysis, the system coefficients can be assumed constant. This assumption was validated by comparative digital simulations.

In Eq. (1), L is defined by

$$L = \mathcal{L}/I_x \tag{6}$$

where $\mathcal{L}$ is the induced roll moment, a strongly nonlinear function of α,β^2;

$$\mathcal{L}(\alpha_T,\phi_A) = f(\alpha_T)\sin 4\phi_A \tag{7}$$

where α_T, the total angle of attack, is defined by

$$\alpha_T = \arccos(\cos\alpha/\cos\beta) \tag{8}$$

and ϕ_A, the aerodynamic roll angle, is defined by

$$\phi_A = \arctan(\tan\beta/\tan\alpha) \tag{9}$$

$f(\alpha_T)$ is roughly parabolic with $f(0)=0$.

Equations (1-5) together with Eqs. (6-9) define the system model. The purpose of this work is to now determine the system behavior when no controls are applied.

The System Absolute Stability

Even after all the assumptions the fifth order system [Eqs. (1-5)] remains strongly nonlinear. The terms pq,pr defining the gyroscopic coupling and $p\beta,p\alpha$ defining the kinematic coupling, together with the induced roll defined in Eq. (7) are still present.

The way this difficulty was overcome in Ref. 1 was to introduce two additional very strong assumptions: 1) The pitch channel is independently controlled assuring a constant α; 2) The induced roll moment is a linear function only of the sideslip angle β. The validity of these assumptions is, however, an open question.

In the present work, only the symmetric characteristics of the missile body will be invoked to prove system stability. For this case

$$M_q = M_r \equiv 2\zeta\omega_n \tag{10}$$

$$M_\alpha = M_\beta \equiv \omega_n^2 \tag{11}$$

$$Z_\alpha = Y_\beta \equiv 1/\tau_\alpha \tag{12}$$

where ω_n is the undamped natural pitch (yaw) frequency; ζ is the pitch (yaw) damping, and τ_a is an equivalent aerodynamic time constant.

As can be seen, we are assuming implicitly that the system coefficients are positive. This in fact implies that the independent pitch, yaw channels are assumed *stable.*

Equations (2-5) can now be rewritten with $\zeta,\omega_n,\tau_\alpha$ and for no control

$$\dot q = 2\zeta\omega_n q + \omega_n^2\alpha + pr \tag{13}$$

$$\dot r = 2\zeta\omega_n r - \omega_n^2\beta - qp \tag{14}$$

$$\dot\alpha = \alpha/\tau_a - \beta p - q \tag{15}$$

$$\dot\beta = -\beta/\tau_a + \alpha p + r \tag{16}$$

Let us now define three new state variables

$$x_1 = r^2 + q^2 \tag{17}$$

$$x_2 = \alpha^2 + \beta^2 \tag{18}$$

$$x_3 = \alpha q - r\beta \tag{19}$$

These new variables are quadratic forms in α,β,r,q. x_1 representing the square of the total transverse body rate and x_2 the square of the total angle of attack.

With these new variables the system is defined by:

$$\dot x_1 = -2\zeta\omega_n x_1 + 2\omega_n^2 x_3 \tag{20}$$

$$\dot x_2 = -2x_2/\tau_a - 2x_3 \tag{21}$$

$$\dot x_3 = -(\zeta\omega_n + 1/\tau_a)x_3 + \omega_n^2 x_2 - x_1 \tag{22}$$

This is a linear system with constant coefficients. Moreover, this system is *independent* of p, the roll rate.

Clearly, the solution of the third order system, Eqs. (20-22), will be only a subset of the previous system solution. However, in what concerns stability, due to the way x_1,x_2 are defined if

$$\lim_{t\to\infty} \|x\| = 0 \tag{23}$$

necessarily implies that α,β,q,r also tend to zero for $t\to\infty$. Moreover, due to the dependence of the induced roll moment α on the total angle of attack, p will also tend to zero for $t\to\infty$. Now, to prove stability for Eqs. (20-22) is a simple matter. The characteristic equation is first obtained. The characteristic equation is of third order and after adequate algebraic manipulations can be shown to be:

$$[s + (2\zeta\omega_n + 1/\tau_a)][s^2 + 2(2\zeta\omega_n + 1/\tau_a)s + 4(\omega_n^2 + 2\zeta\omega_n/\tau_a)] = 0 \tag{24}$$

The main result is as follows: *If the independent linear symmetric pitch, yaw channels are stable, then the entire coupled nonlinear pitch, yaw, roll system is absolutely stable.*

System Behavior

In order to provide a physical insight into the system behavior, we will define the following two, two dimensional vectors:

$$\hat\alpha = (\alpha,\beta) \tag{25}$$

$$\hat r = (r,q) \tag{26}$$

where according to previous definitions:

$$|\hat r| = \sqrt{x_2} \tag{27}$$

$$|\hat\alpha| = \sqrt{x_1} \tag{28}$$

The inner product of these two vectors is given by

$$\hat\alpha\cdot\hat r = \alpha r + \beta q \tag{29}$$

From Eq. (17-19) it can be readily shown that:

$$\hat\alpha\cdot\hat r = (x_1x_2 - x_3^2)^{1/2} \tag{30}$$

which shows that the inner product of $\hat\alpha,\hat r$ is also independent of p. It follows, then, that their entire relative behavior is independent of p. Further, $\hat r$ and $\hat\alpha$ rotate depending on the value of p, but this rotation does not affect their values and their relative position. In particular they can be studied for $p=0$. The presence of p is to be superimposed without modifying their behavior.

This behavior confirms the previously obtained result on the system stability.

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Miss Distance of Proportional Navigation Missile with Varying Velocity

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Introduction

In most of the published work on proportional navigation guidance, the assumption is made that the missile and target velocities remain constant.[1,2] This ensures a collision against the nonmaneuvering target if the missile maintains a constant relative bearing while closing in range. To achieve this constant bearing angle, the proportional navigation missile thus attempts to null the line-of-sight rate.[3]

Axial acceleration or slowdown may seriously influence the performance of the proportional navigation missile.[4] There are two main reasons for this. First, the lateral acceleration and miss distance performance of the missile depend strongly on the effective navigation constant $\bar{N}$, which varies with speed. Velocity compensation is thus usually necessary to ensure that $\bar{N}$ remains within the desired range of 3 to 5.[5] The second direct kinematic effect of nonconstant axial velocity is that it prevents the missile from achieving a constant bearing or rectilinear collision course to the target. An important example here is the short-range engagement with large crossing angle that occurs during a period of large boost acceleration. In this case, however, the magnitude and duration of the boost acceleration are usually known and a simple lateral acceleration correction may be computed and added to the primary guidance signal to compensate for axial acceleration.[6] A more difficult situation may arise at longer ranges and higher altitudes when the missile sustainer motor burns out just before intercept. The missile will normally have limited maneuver capability and a slow dynamic response rate at these high altitudes and, thus, may be unable to entirely compensate for the resulting sudden axial slowdown. A suitable velocity compensation scheme is more difficult in this case. This is because the "time to go" at thrust cutoff and the resulting missile slowdown rate are not precisely known. The missile's heading at cutoff is also unknown. In addition, of course, the absence of any theoretical results on the influence of axial slowdown on miss distance performance further complicates the task of devising a compensation scheme for the above case. The present work provides these results.

The analysis is confined to the noise-free case of a nonmaneuvering crossing air target and considers a single-lag missile having an effective navigation constant of four and unlimited lateral acceleration capability. An approximate analytical solution is developed for the miss distance which consists of two parts. The first is associated directly with the kinematic effects of missile slowdown following sustain thrust cutoff, while the second is due to missile heading error at thrust cutoff. This second component of the miss is already known for the constant-speed case as a linear function of heading error for any integer navigation constant.[1] The task here was to estimate typical thrust cutoff heading errors caused by varying velocity during the sustain thrust phase. This problem is solved using a simplification of the general analysis which neglects higher-order dynamics.

It is shown that if sudden missile slowdown caused by sustain thrust cutoff occurs within three missile time constants of intercept against the high-speed crossing target, the unbiased proportional navigation missile will usually miss behind the target centroid.

As a final note, it is repeated that the present analytical solution for the miss distance due to heading error has been available for a number of years. On the other hand, to the author's knowledge, no previously published solution is available for the miss due to missile acceleration or slowdown.

The Differential Equation

The engagement geometry for the longitudinally accelerating proportional missile against a constant velocity crossing target is shown in Fig. 1. The x axis is inclined to the initial line of sight at ϕ_0, where

$$V_0 \sin\phi_0 = V_T \sin\psi_T \tag{1}$$

Thus, at thrust cutoff, ϕ_0 defines the constant bearing course, or the rectilinear collision course which would be necessary for the missile to intercept the target without further maneuver if the velocity following thrust cutoff were to remain constant at V_0. Following cutoff, however, the linear velocity law

$$V = V_0 + at \tag{2}$$

is assumed, with the missile's initial heading having the small perturbation ϵ relative to ϕ_0. In the following analysis, the quantity ϵ is taken to be zero only when V is assumed to be constant prior to thrust cutoff. The heading error ϵ is estimated later in the section entitled "Missile Heading at the End of the Thrust Phase."

The rate of change of the missile flight-path angle, assuming a single-lag time constant τ, is given by[7]

$$\frac{d\gamma}{dt} = N\left[\frac{d\sigma}{dt} \Big/ 1 + s\tau\right] \tag{3}$$

where $s = d/dt$. Assuming small deviations in γ and σ, Eq. (3) may be written

$$\tau\frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} = NV\frac{d}{dt}\left[\frac{y_T' - y'}{x_T' - x'}\right] \tag{4}$$

where, from Fig. 1 and Eq. (2)

$$x' = \left(V_0 t + \frac{at^2}{2}\right)\cos\phi_0 - y\sin\phi_0 \tag{5}$$

Received Jan. 6, 1984; revision received Oct. 22, 1984.

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